ON THE STABILITY OF CONVECTIVE MOTION OF A BINARY MIXTURE IN A PLANE THERMAL DIFFUSION COLUMN

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B. I. NIKOLAEV and A. A. TUBIN
(Moscow)

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An investigation is carried out with the aid of the Bubnov-Galerkin method of the stability of the stationary convective motion of a binary viscous incompressible mixture in a plane thermal diffusion column with respect to normal perturbations. The spectrum of the decrements is established vs. the gradient of the wave number of the perturbation for Grashof numbers from 0 to 2000. In this motion the existence of monotonic instability is proven. The dependence of the critical Grashof number upon the Prandtl, Schmidt, wave numbers and other parameters is considered. Comparison with the results of experiments is presented.

In [1−4] investigations are to be found of the behavior of normal perturbations in the convective motion of a homogeneous viscous incompressible fluid enclosed between two infinite parallel vertical planes and heated to various temperatures. By means of the Bubnov-Galerkin method the spectrum of decrements was established in the range $0 < kG < 2500$. It was concluded that in the given range of kG there arises a monotonic instability of the steady flow and that oscillatory instability does not occur.

1. In a plane thermal diffusion column of width $2d$ and height $2h\ (h \gg d)$, for a constant temperature difference 2θ between the walls of the binary mixture, with the assumption that $c\,(1-c) = \text{const}$, far from the ends of the column steady distributions of velocity v_0, temperature T_0 and concentration c_0 are established [5] as follows:

$$v_0 = \frac{1}{2a^3(\operatorname{sh}2a + \sin 2a)}\,[\operatorname{sh}a(1+x)\sin a(1-x) - \operatorname{sh}a(1-x)\sin a(1+x)]$$

$$c_0 = -\frac{1}{1+Rs/G}\,x + \frac{1}{a(\operatorname{sh}2a + \sin 2a)}\,[\operatorname{ch}a(1+x)\cos a(1-x) -$$

$$-\operatorname{ch}a(1-x)\cos a(1+x)] + \frac{\varkappa}{G/R+s}\,z, \qquad T_0 = x \qquad (1.1)$$

$$\left(G = \frac{g\beta\theta d^3}{\nu^2},\ R = \frac{g\gamma d^3}{\nu^2},\ \nu = -\frac{1}{\rho}\frac{\partial\rho}{\partial c},\ s = \frac{\alpha c(1-c)\theta}{\langle T\rangle}\right)$$

Here G is the Grashof number, g is the acceleration of gravity, β is the thermal expansion coefficient, ρ is the density, ν is the kinematic viscosity coefficient, α – thermal diffusivity constant and $\langle T\rangle$ is the mean temperature.

The relations (1.1) are presented in nondimensional form. In the present paper the following magnitudes will denote the units of the distance, temperature, velocity, time and concentration, respectively:

$$d,\ \theta,\ \nu/d\ (G+Rs),\ d^2/\nu,\ (G/R+s)$$

The direction of the coordinate axes and the distribution of quantities v_0, c_0 and T_0, for the binary mixture, for $z = 0$, are presented in Fig. 1.

Parameter a, connected with the nondimensional longitudinal concentration gradient $\varkappa$ by the relation

$$a^4 = {}^1/_4\,(G + Rs)\varkappa S \quad (S = \nu / D) \tag{1.2}$$

is defined for the binary liquid mixture by the following equation:

$$\left(5 + \frac{Rs}{G}\right)\frac{1}{2a}\,\frac{\operatorname{ch} 2a - \cos 2a}{\operatorname{sh} 2a + \sin 2a} - 2\left(1 + \frac{Rs}{G}\right)\frac{\operatorname{sh} 2a \sin 2a}{(\operatorname{sh} 2a + \sin 2a)^2} = 2 \tag{1.3}$$

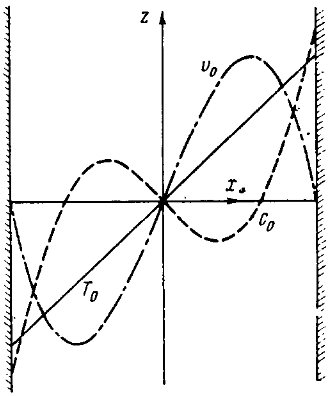

Fig. 1

Here S is the Schmidt number. Let there be generated in the basic flow small two-dimensional perturbations of velocity $\mathbf{u}\ (x, z, t)$, temperature $\vartheta\ (x, z, t)$ and concentration $Q\ (x, z, t)$. Introducing the stream function $\psi\ (x, z, t)$ for the perturbed motion we obtain from the equation of free convection with consideration of thermal diffusion [6] the following boundary value problem:

$$\frac{\partial}{\partial t}\Delta\psi + (G + Rs)\left(v_0 \frac{\partial}{\partial z}\Delta\psi - v_0'' \frac{\partial \psi}{\partial z}\right) = \Delta^2\psi - \frac{G}{G + Rs}\frac{\partial \vartheta}{\partial x} - \frac{\partial Q}{\partial x}$$

$$\frac{\partial \vartheta}{\partial t} + (G + Rs)\left(v_0 \frac{\partial \vartheta}{\partial z} + T_0' \frac{\partial \psi}{\partial z}\right) = \frac{1}{P}\Delta\vartheta$$

$$\frac{\partial Q}{\partial t} + (G + Rs)\left(v_0 \frac{\partial Q}{\partial z} + c_0' \frac{\partial \psi}{\partial z} - x\frac{\partial \psi}{\partial x}\right) = \frac{1}{S}\left(\Delta Q - \frac{Rs}{G + Rs}\Delta\vartheta\right) \tag{1.4}$$

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial z} = \vartheta = \frac{\partial Q}{\partial x} - \frac{Rs}{G + Rs}\frac{\partial \vartheta}{\partial x} = 0 \quad \text{for} \quad x = \pm 1$$

$$\left(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2},\ u_x = \frac{\partial \psi}{\partial z},\ u_z = -\frac{\partial \psi}{\partial x}\right)$$

Here P is the Prandtl number, the prime indicates differentiation with respect to x. The equations (1.4) are linear with respect to ψ, ϑ and Q with coefficients independent of z and t, therefore normal perturbations are considered of the form

$$\psi\ (x, z, t) = \Phi\,(x)\, e^{-\lambda t + ikz}, \qquad \vartheta\,(x, z, t) = T\,(x)\, e^{-\lambda t + ikz}$$
$$Q\,(x, z, t) = N\,(x)\, e^{-\lambda t + ikz} \tag{1.5}$$

Here λ is the complex decrement, $\operatorname{Re}\lambda$ is the parameter for the rate of growth ($\operatorname{Re}\lambda < 0$) or decay ($\operatorname{Re}\lambda > 0$) of the perturbation, $\operatorname{Im}\lambda$ is the frequency of a particular oscillation, k is the real wave number. Substituting (1.5) into (1.4), introducing a new variable

$$C\,(x) = N\,(x) - \frac{Rs}{G + Rs}\,T\,(x)$$

and subtracting from the diffusion equation the equation of heat transfer, multiplied by $Rs/\ (G + Rs)$, we obtain for the amplitudes of perturbations the following equations:

$$L_1\,(\Phi, T, C) \equiv \Delta^2\Phi - ik\,(G + Rs)H\Phi - T' - C' = -\lambda\Delta\Phi$$

$$L_2\,(\Phi, T) \equiv P^{-1}\Delta T - ik\,(G + Rs)\,(v_0 T + T_0'\Phi) = -\lambda T \tag{1.6}$$

$$L_3\,(\Phi, T, C) \equiv S^{-1}\Delta C - ik(G + Rs)\left[v_0 C + \left(c_0' - \frac{Rs}{G + Rs}T_0'\right)\Phi\right] + (G + Rs)\,\varkappa\Phi' - \frac{Rs}{(G + Rs)P}\Delta T = -\lambda C$$

$$\Phi(\pm 1) = \Phi'(\pm 1) = T(\pm 1) = C'(\pm 1) = 0 \qquad (1.7)$$

$$\left(\Delta = \frac{\partial^2}{\partial x^2} - k^2, \ H = v_0 \Delta - v_0'' \right)$$

2. The eigenvalue problem (1.6), (1.7) is solved by the Bubnov-Galerkin method. An approximate solution of the problem will be sought in the form of the expansions

$$\Phi^* = \sum_{m=0}^{p-1} a_m \varphi_m^{(0)}, \quad T^* = \sum_{r=0}^{q-1} b_r \theta_r^{(0)}, \quad C^* = \sum_{j=0}^{w-1} l_j \xi_j^{(0)} \qquad (2.1)$$

Here the coordinate functions $\varphi_m^{(0)}$, $\theta_r^{(0)}$ and $\xi_j^{(0)}$ represent the eigenfunctions of the operators [7] related to the operators (1.6)

$$\Delta^2 \varphi_m^{(0)} + \chi_m^{(0)} \Delta \varphi_m^{(0)} = 0, \qquad \varphi_m^{(0)}(\pm 1) = \varphi_m^{(0)\prime}(\pm 1) = 0 \qquad (2.2)$$

$$P^{-1} \Delta \theta_r^{(0)} + \omega_r^{(0)} \theta_r^{(0)} = 0, \qquad \theta_r^{(0)}(\pm 1) = 0 \qquad (2.3)$$

$$S^{-1} \Delta \xi_j^{(0)} + \mu_j^{(0)} \xi_j^{(0)} = 0, \qquad \xi_j^{(0)\prime}(\pm 1) = 0 \qquad (2.4)$$

Problem (2.2) has even and odd solutions. The even eigenfunctions are:

$$\varphi_m^{(0)} = \frac{\operatorname{ch} kx}{\operatorname{ch} k} - \frac{\cos \sqrt{\chi_m^{(0)} - k^2} x}{\cos \sqrt{\chi_m^{(0)} - k^2}} \qquad (m = 0, 2, 4, \ldots) \qquad (\chi_m^{(0)} > k^2) \qquad (2.5)$$

The decrements of the even perturbations $\chi_m^{(0)}$ are derived from the characteristic equation

$$k \operatorname{th} k + \sqrt{\chi_m^{(0)} - k^2} \operatorname{tg} \sqrt{\chi_m^{(0)} - k^2} = 0 \qquad (2.6)$$

The odd eigenfunctions $\varphi_m^{(0)}$ have the form

$$\varphi_m^{(0)} = \frac{\operatorname{sh} kx}{\operatorname{sh} k} - \frac{\sin \sqrt{\chi_m^{(0)} - k^2} x}{\sin \sqrt{\chi_m^{(0)} - k^2}} \qquad (m = 1, 3, 5, \ldots) \qquad (2.7)$$

The spectrum of eigenvalues of the odd perturbations $\chi_m^{(0)}$ is derived from the equation

$$k \operatorname{cth} k - \sqrt{\chi_m^{(0)} - k^2} \operatorname{ctg} \sqrt{\chi_m^{(0)} - k^2} = 0 \qquad (2.8)$$

Equations (2.2) yield the condition of orthogonality

$$\int_{-1}^{1} \varphi_n^{(0)} \Delta \varphi_m^{(0)} dx = 0 \qquad (n \neq m) \qquad (2.9)$$

The spectrum of eigenvalues $\omega_r^{(0)}$ and the amplitudes of the temperature perturbations $\theta_r^{(0)}$, are found for problem (2.3)

$$\omega_r^{(0)} = P^{-1} [{}^1/_4 \pi^2 (r+1)^2 + k^2] \qquad (r = 0, 1, 2, \ldots)$$

$$\theta_r^{(0)} = \begin{cases} \cos {}^1/_2 \pi (r+1) x & (r = 0, 2, 4, \ldots) \\ \sin {}^1/_2 \pi (r+1) x & (r = 1, 3, 5, \ldots) \end{cases} \qquad (2.10)$$

From the symmetry of the operator (2.3) there follows the orthogonality of the temperature perturbations

$$\int_{-1}^{1} \theta_r^{(0)} \theta_c^{(0)} dx = 0 \qquad (r \neq c) \qquad (2.11)$$

The spectrum of decrements $\mu_j^{(0)}$ and of the normalized perturbations of concentration $\xi_j^{(0)}$ are found from the boundary value problem (2.4)

$$\mu_j^{(0)} = S^{-1}(\tfrac{1}{4}\pi^2 j^2 + k^2) \qquad (j = 0, 1, 2, \ldots)$$

$$\xi_j^{(0)} = \begin{cases} \tfrac{1}{2}\sqrt{2} & (j = 0) \\ \sin \tfrac{1}{2}\pi j x & (j = 1, 3, 5, \ldots) \\ \cos \tfrac{1}{2}\pi j x & (j = 2, 4, 6, \ldots) \end{cases} \tag{2.12}$$

The perturbations of concentration are orthogonal

$$\int_{-1}^{1} \xi_j^{(0)} \xi_h^{(0)}\, dx = 0 \qquad (j \neq h) \tag{2.13}$$

In the case of small temperature difference of the walls of the thermodiffusion column, the solution of the boundary value problem (1.6), (1.7) can be sought in the form of expansion in a series of the small parameter $ik\,(G + Rs)$. In this case it can be shown that the problems (2.2)–(2.4) determine the spectra of decrements $\chi_m^{(0)}$, $\omega_r^{(0)}$ and $\mu_j^{(0)}$ of the perturbations in the quiescent liquid when the walls of the thermodiffusion column are held at the same temperature. In the same manner as in [3] the decrements $\chi_m^{(0)}$ define isothermal perturbations in the absence of perturbations of concentration. The spectrum of the decrements $\mu_j^{(0)}$ define the isothermal concentration perturbations in the binary mixture and $\omega_r^{(0)}$ define the nonisothermal perturbations.

The analysis of the systems of equations obtained from (1.6) by expansion of the perturbation amplitude (1.5) and of the decrement λ in the small parameter $ik\,(G + Rs)$ shows that all odd corrections for the decrements of zero approximation $\chi_m^{(0)}$, $\mu_j^{(0)}$ and $\omega_r^{(0)}$, are identically zero. Hence the series of eigenvalue λ contains only even exponents of the small parameter $ik\,(G + Rs)$, i.e. λ is real. In this manner, for small values of the Grashof number in the thermodiffusion column there arise monotonic perturbations with zero phase velocity ("standing" perturbations).

3. According to the Bubnov-Galerkin method we require the orthogonality of the functions $L_1\,(\Phi^*, T^*, C^*)$, $L_2\,(\Phi^*, T^*)$, $L_3\,(\Phi^*, T^*, C^*)$ respectively to the functions $\varphi_m^{(0)}$ $(m = 0, 1, 2, \ldots, p-1)$, $\theta_r^{(0)}$ $(r = 0, 1, 2, \ldots, q-1)$, $\xi_j^{(0)}(j = 0, 1, 2, \ldots, w-1)$. This leads to the system of linear homogeneous algebraic equations

$$\sum_{m=0}^{p-1} a_m\,[(\chi_m^{(0)} - \lambda)\,\delta_{nm} + ik(G + Rs)\,H_{nm}] + \sum_{r=0}^{q-1} b_r A_{nr} + \sum_{j=0}^{w-1} l_j B_{nj} = 0$$
$$(n = 0, 1, 2, \ldots, p-1)$$

$$ik\,(G + Rs) \sum_{m=0}^{p-1} a_m E_{cm} + \sum_{r=0}^{q-1} b_r\,[(\omega_r^{(0)} - \lambda)\,\delta_{cr} + ik\,(G + Rs)\,D_{cr}] = 0 \tag{3.1}$$
$$(c = 0, 1, 2, \ldots, q-1)$$

$$\sum_{m=0}^{p-1} a_m\,[ik\,(G + Rs)\,W_{hm} - \varkappa\,(G + Rs)\,Z_{hm}] + \frac{Rs}{(G + Rs)P} \sum_{r=0}^{q-1} b_r \Pi_{hr} +$$
$$+ \sum_{j=0}^{w-1} l_j\,[(\mu_j^{(0)} - \lambda)\,\delta_{hj} + ik\,(G + Rs)\,V_{hj}] = 0 \quad (h = 0, 1, 2, \ldots, w-1)$$

Here δ_{ij} is the Kronecker operator

$$H_{nm} = \frac{1}{F_n}\int \varphi_n^{(0)} H \varphi_m^{(0)} dx$$

$$F_n = \int \varphi_n^{(0)} \Delta \varphi_n^{(0)} dx = \begin{cases} \chi_n^{(0)} [(\chi_n^{(0)} - k^2)^{-1} k \operatorname{th} k (1 - k \operatorname{th} k) - 1] & (n = 0, 2, 4, \ldots) \\ \chi_n^{(0)} [(\chi_n^{(0)} - k^2)^{-1} k \operatorname{cth} k (1 - k \operatorname{cth} k) - 1] & (n = 1, 3, 5, \ldots) \end{cases}$$

$$A_{nr} = \frac{1}{F_n}\int \varphi_n^{(0)} \theta_r^{(0)\prime} dx, \qquad B_{nj} = \frac{1}{F_n}\int \varphi_n^{(0)} \xi_i^{(0)\prime} dx \tag{3.2}$$

$$E_{cm} = \frac{1}{Y_c}\int \theta_c^{(0)} T_0' \varphi_m^{(0)} dx, \quad Y_c = \int \theta_c^{(0)2} dx = 1, \quad D_{cr} = \frac{1}{Y_c}\int \theta_c^{(0)} v_0 \theta_r^{(0)} dx$$

$$W_{hm} = \frac{1}{I_h}\int \xi_h^{(0)} \left(c_0' - \frac{Rs}{G + Rs} T_0'\right) \varphi_m^{(0)} dx, \qquad I_h = \int \xi_h^{(0)2} dx = 1$$

$$Z_{hm} = \frac{1}{I_h}\int \xi_h^{(0)} \varphi_m^{(0)\prime} dx, \quad \Pi_{hr} = -\frac{1}{I_h}\int \xi_h^{(0)} \Delta \theta_r^{(0)} dx, \quad V_{hj} = \frac{1}{I_h}\int \xi_h^{(0)} v_0 \xi_j^{(0)} dx$$

(the integrals extend between the limits ± 1).

Explicit form of the matrix elements are not presented here because they are too cumbersome.

The conditions of existence of the nontrivial solution of the system (3.1) offers the possibility of calculating the eigenvalues $\lambda = \lambda(G, P, S, R, s, k)$. This problem reduces to the determination of the eigenvalues of the complex matrix $(p + q + w)$ consisting of the elements (3.2). The spectrum of the perturbation decrements λ was determined by the orthogonal-exponent method on a computer.

4. The computation of the decrements was carried out by using 30 coordinate functions for the quantities p, q and w, varied from 8 to 12 in dependence upon the Prandtl and Schmidt numbers. As a result, with the approximation introduced, stable eigenvalues of 30 lower levels were obtained in the range $0 < kG < 2000$. As in [4], with the increase of the Grashof number the convergence of the series (2.1) became worse. In the given case the convergence was verified for the decrements computed with 27, 30 and 33 approximations. Agreement was obtained with sufficient accuracy with 30 and 33 coordinate functions used.

In Figs. 2 and 3 are presented the functional dependence of $\sqrt{\operatorname{Re}\lambda}$ upon the parameter $\sqrt{G}$ for $k = 1, S = 3, s = 5\times 10^{-3}, R = 200$ for two values of the Prandtl number $P = 1$ and $P = 0.1$ (solid lines are for χ_m levels, dashed lines for ω_r levels, dotted lines for μ_j levels and dot-dashed lines show the real part of the complex conjugate decrements).

For small Grashof numbers all decrements are real and positive and consequently, all perturbations decay monotonically. With increasing Grashof number, there arises a mutual interaction of the real levels with the development of the complex conjugate pairs (two perturbances propagating with identical but opposite phase velocity). The phase velocity of the oscillatory perturbations increases noticeably with increase of the G number. The primary merging of the χ_m and ω_r is observed; the complex conjugate pairs $\mu_j - \chi_m$ and $\mu_j - \omega_r$ rapidly decay with increase of G and, subsequently μ_j merge between themselves only.

For sufficiently large value of G, the split of any complex conjugate pair $\chi_m - \omega_r$ on two real levels is observed. One of these levels in due course intersects the axis G ; this leads to monotonic instability of the basic flow.

By analogy with [4] there exists a definite dependence of the eigenvalues spectrum on the Prandtl and Schmidt numbers. Thus, for $P = 0.1$ (Fig. 3) below χ_0 level, the ω_r levels are absent and decrement χ_0 intersects the axis G independently of presence of μ_j levels in this region. This results in instability of the steady flow.

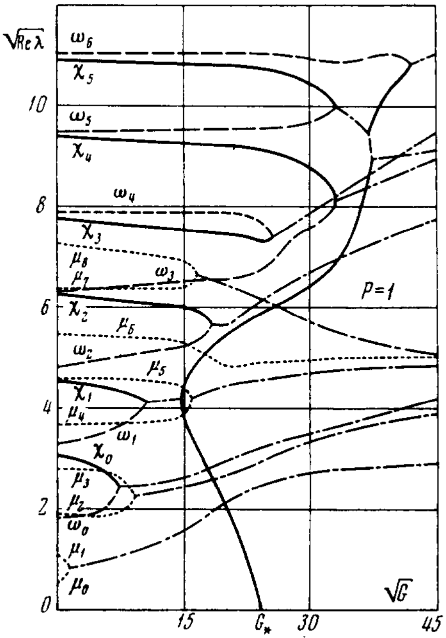

Fig. 2

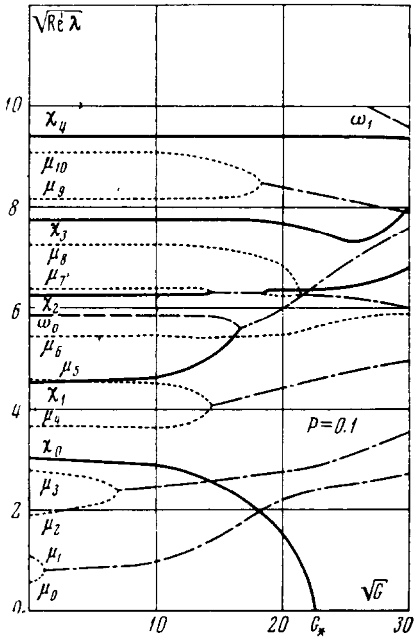

Fig. 3

5. In Figs. 2 and 3, the point of intersection of the axis G by real level determines the critical Grashof number G_* ($k = 1$) separating the region of stable values of G from the unstable values with respect to "standing" perturbations. In Fig. 4a are plotted the neutral curves $k = k(\sqrt{G_*})$ (Imλ=0, $0<kG<2000$) for two values of the Prandtl number: Curve *1* for $P = 5$ and Curve *2* for $P = 0.1$. As was to be expected, instability of the basic flow is generated by perturbations with large wavelengths. The smallest wave length for unstable perturbations equals $3.5\,d$. The wave number at which the smallest critical Grashof number is attained is approximately 1.4 and it varies but slightly with

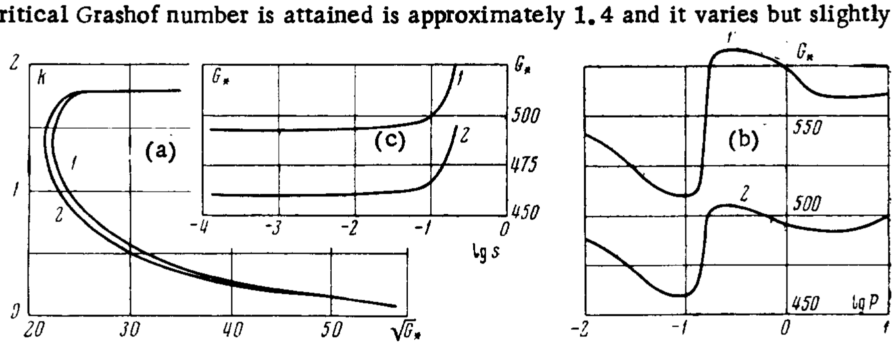

Fig. 4

Schmidt number in the range $0.01 < S < 10$ and with Prandtl number in the range $0.01 < P < 10$. This agrees with the findings of [4].

In Fig. 4b are plotted the critical Grashof number as a function of the logarithm of Prandtl number for two values of the wave number: Curve *1* for $k = 1$, Curve *2* for $k = 1.4$. The variation of the critical Grashof number amounts to not more than 1% for a variation of Schmidt number in the range $0.01 < S < 10$. Within the given approximation it does not appear possible to investigate the spectra of decrements for $P > 10$ or $S > 10$ for reasons discussed in [4].

In Fig. 4c is plotted the minimum critical Grashof number as a function of the thermodiffusion parameter s for two values of Prandtl number: Curve *1* for $P = 5$ and Curve *2* for $P = 0.1$. An increase of the thermodiffusion constant brings about some increase of the stability of the basic flow. A corresponding relationship exists for the variation of R in the range $1 < R < 10^4$.

The weak dependence of the critical Grashof number upon the Prandtl and Schmidt numbers, the thermodiffusion parameter s and upon R and also the character of behavior of the spectra of decrements justify the conclusion that the instability of the convection current of a binary mixture in a thermodiffusion column is related to the hydrodynamic instability of the interface of the opposite convective currents [8].

In [9] an experimental investigation has been presented of the instability of laminar flow in a thermodiffusion column, carried out with a gas of $P \approx 0.8$. The experiment based upon the recognition of the critical nature of heat transfer across a layer of gas has shown that convective flow is stable up to $G_* = 585$, although it does not appear possible to draw a definite conclusion in regard to the nature of the instability encountered in this kind of experiment. The theoretical calculation carried out for the same Prandtl number gave a minimum critical Grashof number pertaining to "standing" perturbations $G_* = 505$.

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